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MODELING OF VERTICAL PROPAGATION OF ATMOSPHERIC ACOUSTIC DISTURBANCE INITIATED BY A PULSE AT THE LOWER BOUNDARY

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Initial equations

System of hydrothermodynamics

$$\frac{\partial \vec{V}}{\partial t} = -\frac{\vec{\nabla} p'}{\overline{\rho}} + \vec{g}\frac{\rho'}{\overline{\rho}}, \tag{1}$$

$$\frac{\partial p'}{\partial t} = -\vec{V} \cdot (\vec{\nabla} \overline{p}) - \gamma \overline{p} (\vec{\nabla} \cdot \vec{V}), \tag{2}$$

$$\frac{\partial \rho'}{\partial t} = -\vec{V} \cdot (\vec{\nabla} \overline{\rho}) - \overline{\rho} (\vec{\nabla} \cdot \vec{V}), \tag{3}$$

where $\vec{V}$ is the gas flow velocity; p' is the wave contribution to the background pressure $\overline{p}$; ρ' is the wave contribution to the background density $\overline{\rho}$.

$$\bar{\rho}(z) = \rho_0 \exp\left(-\frac{z}{H}\right), \quad \bar{p}(z) = p_0 \exp\left(-\frac{z}{H}\right) = \bar{\rho}(z) g H, \tag{4}$$

where $\rho_0 = \bar{\rho}(z=0),\ p_0 = \bar{p}(z=0)$, and the relationship between the equilibrium pressure and density follows from the stationary zero-order equality $d\bar{p}/dz = -g\bar{\rho}(z)$, given $g_x = 0, g_y = 0, g_z = -g$.

Initial equations

1D system of hydrothermodynamics

The one-dimensional linearized system of hydrothermodynamics:

$$\frac{\partial U}{\partial t} = \frac{1}{\rho_0}\left(\frac{\gamma-2}{2\gamma H(0)} - \frac{H(z)}{H(0)}\frac{\partial}{\partial z}\right) P + \frac{\Phi}{\gamma H(0)\rho_0}, \tag{5}$$

$$\frac{\partial P}{\partial t} = -\gamma g H(0)\rho_0 \frac{\partial U}{\partial z} - gH(0)\rho_0 \frac{\gamma-2}{2H(z)} U, \tag{6}$$

$$\frac{\partial \Phi}{\partial t} = -\frac{\gamma - 1 + \gamma\frac{dH(z)}{dz}}{H(z)} gH(0)\rho_0 U, \tag{7}$$

Here P, Φ, U are functions that represent the pressure perturbation p', the entropy perturbation $\varphi' = p' - \gamma\rho'\overline{p}/\overline{\rho}$ and the vertical flow velocity V respectively and are related to the real values as

$$P = p' \cdot \exp\left(\int\limits_0^z \frac{dz'}{2H(z')}\right), \; \Phi = \varphi' \cdot \exp\left(\int\limits_0^z \frac{dz'}{2H(z')}\right), \; U = V \cdot \exp\left(-\int\limits_0^z \frac{dz'}{2H(z')}\right),$$

$\gamma = C_p/C_v$; C_p, C_v are molar heat capacities at constant pressure and volume, $g = g_z$ is the vertical component of gravity field vector $\vec{g}$, ρ_0 and ρ' is the air density at the lower boundary its perturbation respectively.

Initial equations

1D Klein-Gordon equation

The one-dimensional system of hydrothermodynamics (1)-(3) can be reduced to the one-dimensional Klein-Gordon equation by differentiating the equation (5) with respect to the time t and replacing the derivatives $\partial P/\partial t$ and $\partial \Phi/\partial t$ with (6) and (7):

$$\frac{\partial^2 U}{\partial t^2} - \gamma g H(z)\frac{\partial^2 U}{\partial z^2} + \frac{g\gamma}{4H(z)}\left(1 + 2\frac{dH(z)}{dz}\right)U = 0 \tag{8}$$

or

$$\frac{\partial^2 U}{\partial t^2} - c^2(z)\frac{\partial^2 U}{\partial z^2} + a(z)U = 0, \tag{9}$$

where

$$c(z) = \sqrt{\gamma g H(z)}, \quad a(z) = \frac{g\gamma}{4H(z)}\left(1 + 2\frac{dH(z)}{dz}\right). \tag{10}$$

Initial equations

Initial and boundary condition

We complete Klein-Gordon equation with the initial-boundary conditions:

$$U\Big|_{t=0} = U_t\Big|_{t=0} = 0,\ U\Big|_{z=0} = F^n(t), \tag{11}$$

$$F^n(t) = \frac{A_n}{(n+1)!}\lambda^{n+2}t^{n+1}e^{-\lambda t}, \quad \text{at} \quad t > 0, \quad F(t) = 0, \quad \text{at} \quad t \leq 0, \tag{12}$$

where λ characterizes the duration of the pulse and A its amplitude.
Later we will discuss results obtained with the boundary conditions for $n = 0, 1, 2$:

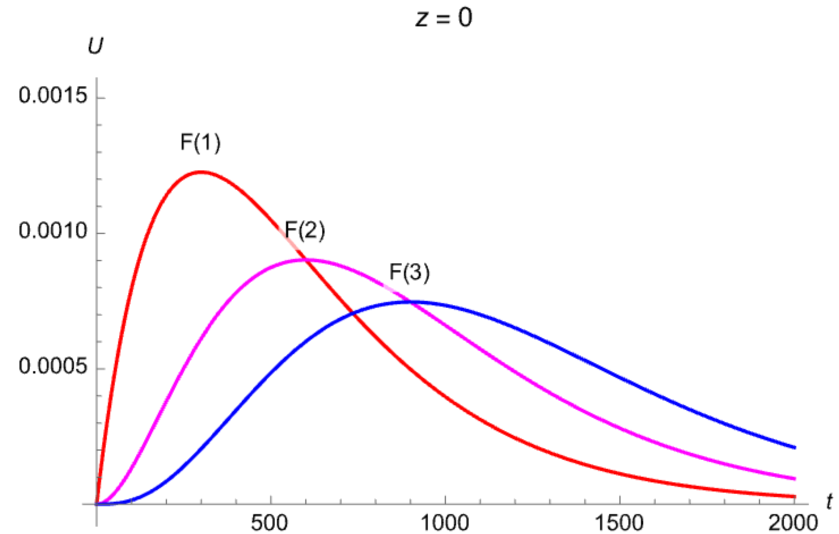

Figure: Boundary conditions $F^1(t) = A\lambda^2 te^{-\lambda t}$ (red), $F^2(\tau) = \frac{A}{2}\lambda^3 t^2 e^{-\lambda t}$ (pink), $F^3(\tau) = \frac{A}{6}\lambda^4 t^3 e^{-\lambda t}$ (blue) for $A = 1, \lambda = 1/300$.

Solution

Analytical solution in case $H = H_0$

Theorem 1. For constant value of the atmospheric scale height $H(z) = H(z=0) = H_0$ the coefficients take on a constant value $c(0), a(0)$ and the initial-boundary value problem for the Klein-Gordon equation can be analytically solved [1]:

$$U(t,z) = U_1(t,z) + U_2(t,z) \tag{13}$$

$$U_1(t,z) = \frac{1}{\pi}\mathrm{Re}\int\limits_0^{\sqrt{a(0)}} \mathcal{F}(i\eta) e^{\left(i\eta t - \frac{z}{c(0)}\sqrt{a(0)-\eta^2}\right)} d\eta, \tag{14}$$

$$U_2(t,z) = \frac{1}{\pi}\mathrm{Re}\int\limits_{\sqrt{a(0)}}^{\infty} \mathcal{F}(i\eta) e^{i\left(\eta t - \frac{z}{c(0)}\sqrt{\eta^2-a(0)}\right)} d\eta, \tag{15}$$

where $\mathcal{F}(i\eta)$ is the Laplace image of the boundary condition (11):

$$\mathcal{F}(i\eta) = \mathcal{F}(s) = \int_0^\infty F(t)e^{-st}\, dt. \tag{16}$$

Remark 1: For $a = 0$, the resulting formula (13) becomes the exact formula of the initial-boundary value problem solution for the wave equation.

[1] Smirnova, E.S. *Asymptotics of the Solution of an Initial–Boundary Value Problem for the One-Dimensional Klein–Gordon Equation on the Half-Line.* Math Notes 114, 608–618 (2023). https://doi.org/10.1134/S0001434623090286

Solution

H=H(z) approximation

Since finding a solution in the case of $H = H(z)$ requires its explicit form, for the altitude range from 0 to 200 km, the following approximation was chosen:

$$H(z) = 7000 + 0.135z\tanh^5\left(\frac{z}{12000}\right). \quad (17)$$

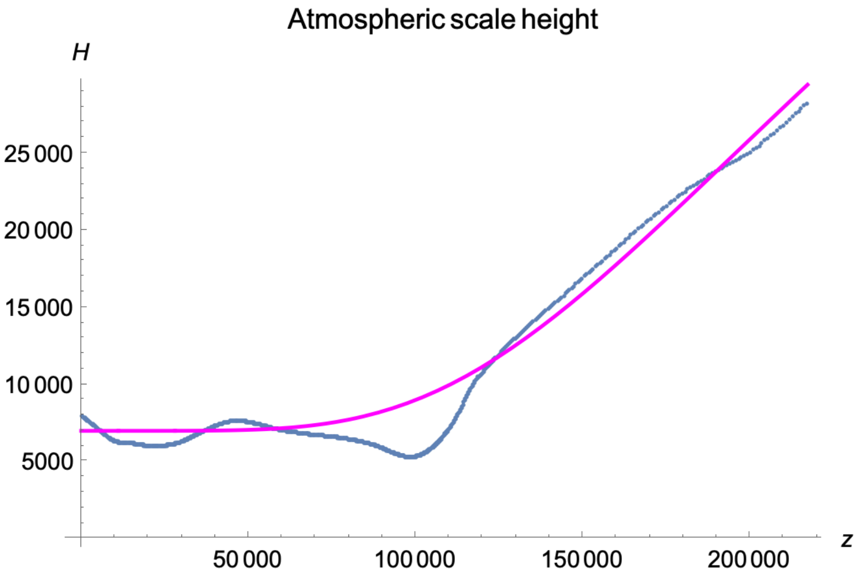

Figure: Atmospheric scale height $H = H(T(z))$ calculated from numerical simulation data for temperature (blue) and its approximation (17) (pink).

Remark 2: Such an approximation at low altitudes gives an almost constant value of both the atmospheric scale height $H = H(z)$ and of the main coefficients $c(z)$ and $a(z)$ of the Klein-Gordon equation. Thus, in this area the solution of the initial boundary value problem in the case of variable coefficients $c(z)$ and $a(z)$ coincides with the exact solution for constant values of the coefficients $c(0)$ and $a(0)$.

Solution

Asymptotic solution in case $H = H(z)$

Theorem 2: The leading term of the formal asymptotic solution of the problem (9),(11) is determined by the formulas [2]:

$$U = U_1 + U_2, \quad U_1(\tau, y) = \frac{e^{-\frac{y}{h}}}{\pi\tau}\mathrm{Re}\left[\mathcal{F}(0,h)\right], \quad U_2 = \frac{c(y)}{c(0)}\left(K_{\Lambda^+_\tau}A^+_0 + K_{\Lambda^-_\tau}A^-_0\right). \tag{18}$$

Remark 3: For low altitudes in the region under study, the form of the asymptotic of the wave-propagating part of the solution:

$$U_2(y,\tau) = K_{\Lambda^+_\tau}A^+_0 + K_{\Lambda^-_\tau}A^-_0 = \frac{1}{2\pi h}\int\limits_0^\infty \overline{\mathcal{F}}(\sqrt{c^2(0)p^2 + a(0)},\, h)e^{\frac{i}{h}\left(py - \tau\sqrt{c^2(0)p^2+a(0)}\right)}\frac{c^2(0)p}{\sqrt{c^2(0)p^2 + a(0)}}dp -$$
$$-\frac{1}{2\pi h}\int\limits_{-\infty}^0 \mathcal{F}(\sqrt{c^2(0)p^2 + a(0)},\, h)e^{\frac{i}{h}\left(py + \tau\sqrt{c^2(0)p^2+a(0)}\right)}\frac{c^2(0)p}{\sqrt{c^2(0)p^2 + a(0)}}dp = \tag{19}$$
$$= \frac{1}{\pi h}\mathrm{Re}\int\limits_0^\infty \mathcal{F}(\sqrt{c^2(0)p^2 + a(0)}, h)e^{\frac{i}{h}\left(\tau\sqrt{c^2(0)p^2+a(0)} - py\right)}\frac{c^2(0)p}{\sqrt{c^2(0)p^2 + a(0)}}dp.$$

For medium and high altitudes in the region under study, the form of the asymptotics of the wave-propagating part of the solution:

$$U_2 = \frac{c(y)}{c(0)}\left(K_{\Lambda^+_\tau}A^+_0 + K_{\Lambda^-_\tau}A^-_0\right) = \frac{1}{\sqrt{2\pi h}}\frac{c(y)}{c(0)}\left(\frac{e^{\frac{i}{h}S^+(\alpha,\tau)}\alpha c^2(0)}{\sqrt{J^+(\alpha,\tau)(\alpha^2c^2(0) + a(0))}}\bar{\mathcal{F}}(\sqrt{\alpha^2c^2(0) + a(0)}, h)\right)\Bigg|_{\alpha=\alpha^+(y,\tau)} +$$
$$+\frac{1}{\sqrt{2\pi h}}\frac{c(y)}{c(0)}\left(\frac{e^{-\frac{i}{h}S^-(\alpha,\tau)}\alpha c^2(0)}{\sqrt{J^-(\alpha,\tau)(\alpha^2c^2(0) + a(0))}}\mathcal{F}(\sqrt{\alpha^2c^2(0) + a(0)}, h)\right)\Bigg|_{\alpha=\alpha^-(y,\tau)} = \tag{20}$$
$$= \sqrt{\frac{2}{\pi h}}\frac{c(y)}{c(0)}\mathrm{Re}\left(\frac{e^{\frac{i}{h}S^+(\alpha,\tau)}\alpha c^2(0)}{\sqrt{J^+(\alpha,\tau)(\alpha^2c^2(0) + a(0))}}\bar{\mathcal{F}}(\sqrt{\alpha^2c^2(0) + a(0)}, h)\right)\Bigg|_{\alpha=\alpha^+(y,\tau)}.$$

[2] Dobrokhotov, S., Smirnova, E. *Asymptotics of the Solution of the Initial Boundary Value Problem for the One-Dimensional Klein–Gordon Equation with Variable Coefficients*. Russ. J. Math. Phys. 31, 187–198 (2024).

Results

Asymptotic solution in case $H = H_0$

Return to the physical functions of flow velocity disturbance is carried out through (8):

$$V = U \cdot \exp\left(\frac{z}{2H_0}\right) \tag{21}$$

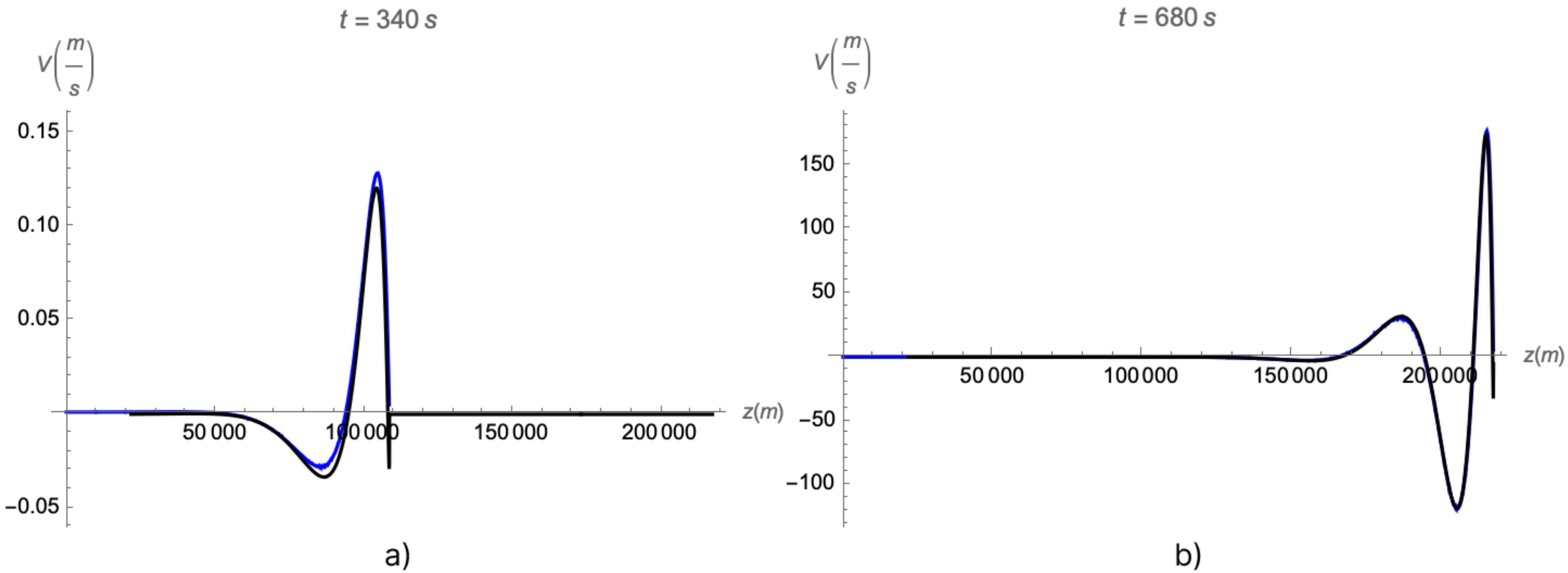

Figure: Comparison of analytical (numerical calculation) and asymptotic solutions for constant value $H = H_0 = 7000$ m.

Results

Asymptotic solution in case $H = H(z)$

Return to the physical functions of flow velocity disturbance is carried out through (8):

$$V = U \cdot \exp\left(\int_0^z \frac{dz'}{2H(z')}\right) \tag{22}$$

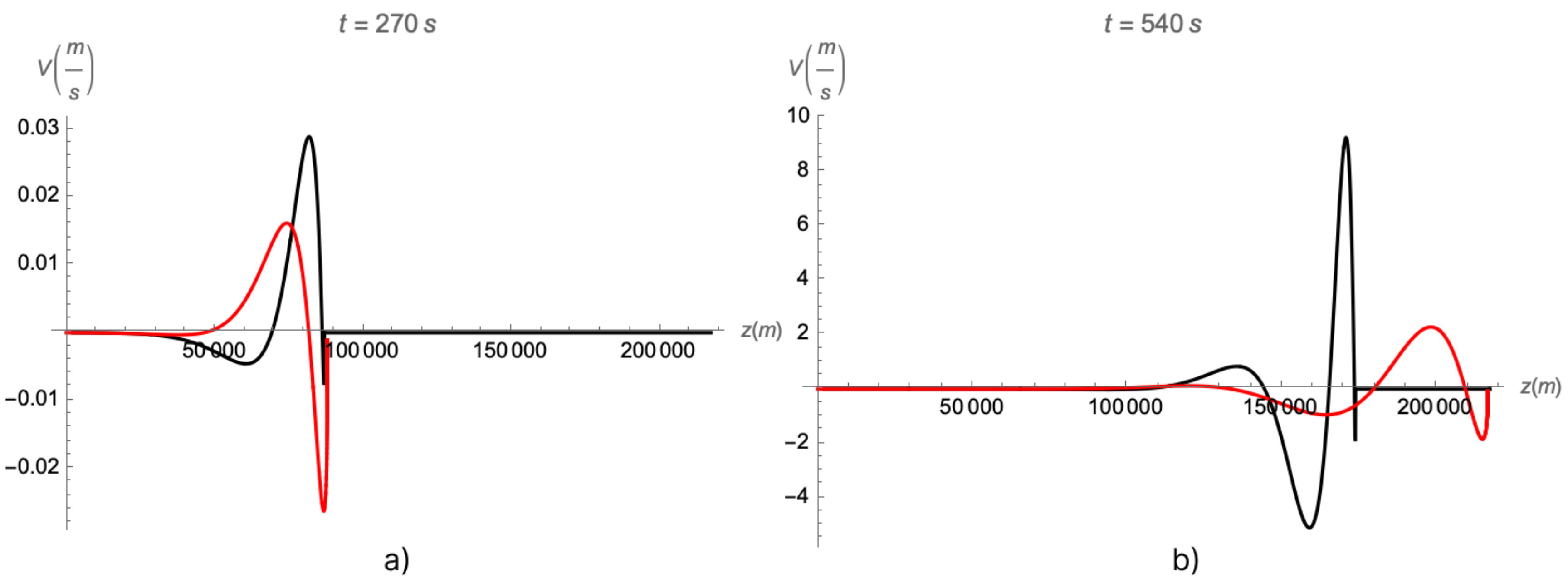

Figure: Comparison of time evolution of the asymptotic solutions (19) for $H = H_0 = 7000$ m (black) and (20) for $H = H(z)$ (17) (red).

Results

Different boundary conditions

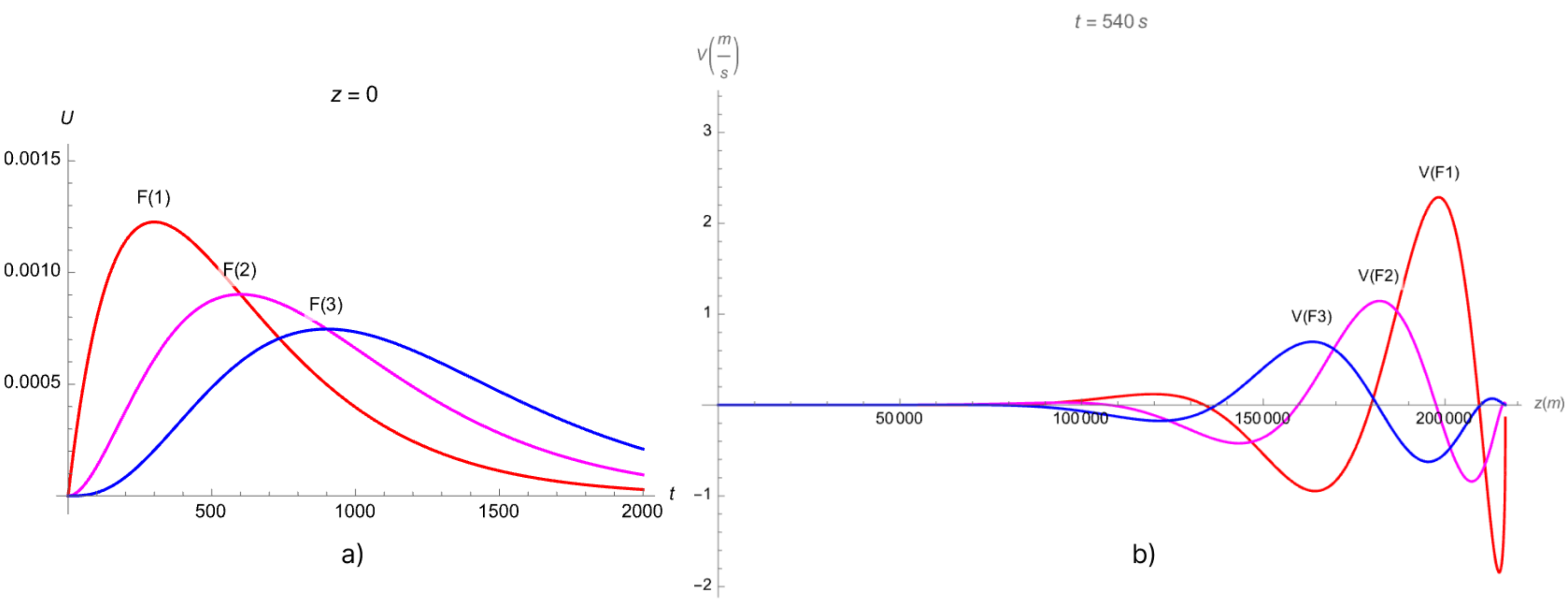

Figure: **(a)** Boundary conditions $F^1(t)$ (red), $F^2(t)$ (pink), $F^3(t)$ (blue); **(b)** Asymptotics of solution at $t = 540$ s in case $H = H(z)$ corresponding each boundary condition. To simplify the analysis, the amplitudes were normalized: $A_1 = 1$; $A_2 = 10$; $A_3 = 100$.

Discussion

The initial-boundary value problem is solved both analytically and asymptotically in a general form, therefore, parameters of the problem, as well as the boundary condition, can be refined for a more specific physical problem.

Thank you for your attention.