Ballooning instability and conditions for it development the Earth's magnetosphere

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1 Introduction: Ballooning instability

- It develops in the magnetosphere, taking into account the finite plasma pressure and curvature of the lines of force;
- Provided that plasma pressure drops rapidly with distance from the Earth;
- For azimuthally small-scale modes $(\lambda_a \ll \lambda_{\|})$;
- Can be responsible for initiating sub-storms in the magnetosphere.

2 Introduction: Unsolved problems

What was studied:

- Transition of Alfvén modes to the ballooning instability mode (Oberhagemann et. al., 2020);
- Transition of slow magnetosonic modes (SMS) to the ballooning instability mode (Mazur et al., 2013)

What was not studied:

- Connection between ballooning instability and MHD modes in non-uniform plasma (Alfvén, SMS);
- Dependence of development conditions for instability on the radial wavelength (usually study in the limit $k_r/k_a \approx 0$)

3 Problem statement

- Determine the conditions for the development of ballooning instability depending on the plasma pressure and its gradient for the first two harmonics.
- To study the dispersion relation of coupled Alfvén and SMS modes. To obtain the numerical dependence of the γ instability growth rate on the ratio of the radial and azimuthal components of the wave vector k_r/k_a for a two-dimensionally inhomogeneous magnetosphere model.
- To study the longitudinal structure of eigenfunctions of coupled Alfvén and SMS modes for the first two harmonics in stable and unstable regimes.

4 Basic equations

$$\frac{k_1^2}{k_2^2}\hat{L}_T(\omega)\Phi + \hat{L}_P(\omega)\Phi - \sqrt{\frac{g_1}{g_2}}\frac{8\pi}{B^2 R}\frac{\partial P}{\partial L}\Phi = \sqrt{g_1}\frac{2\omega}{R}\frac{V_S}{V_A}\Theta, \qquad (1)$$

$$\hat{L}_S(\omega)\Theta = \sqrt{g_1}\frac{2\omega}{R}\frac{V_S}{V_A}\Phi. \qquad (2)$$

Φ — transverse potential of the electric field:

$$E = E_\perp = -\nabla_\perp \Phi \qquad (3)$$

uniform plasma:
Φ — Alfvén mode
Θ — SMS mode
non-uniform plasma:
Φ — transverse electric field of the wave
Θ — rarefaction/compression of plasma

Θ —proportional to the divergence of the plasma displacement:

$$\Theta = \frac{\sqrt{4\pi\gamma P}}{c k_a}\nabla\cdot\boldsymbol{\xi} \qquad (4)$$

Differential operators:

$$\hat{L}_T(\omega) = \frac{\partial}{\partial l_\parallel}\sqrt{\frac{g_2}{g_1}}\frac{\partial}{\partial l_\parallel} + \sqrt{\frac{g_2}{g_1}}\frac{\omega^2}{V_A^2}, \qquad (5)$$

$$\hat{L}_P(\omega) = \frac{\partial}{\partial l_\parallel}\sqrt{\frac{g_1}{g_2}}\frac{\partial}{\partial l_\parallel} + \sqrt{\frac{g_1}{g_2}}\frac{\omega^2}{V_A^2}, \qquad (6)$$

Characteristics of the ionosphere:

Ideal conductivity:

$$\Phi|_{\theta=\theta_\pm} = 0. \tag{7}$$

High density, $\xi_\parallel = 0$:

$$\left.\frac{\partial\Theta}{\partial l_\parallel}\right|_{\theta=\theta_\pm} = 0. \tag{8}$$

$$\frac{\omega^2}{V_S^2}\frac{\omega}{k_a c} = -\frac{\omega}{\sqrt{4\pi\gamma P}}\frac{\partial\Theta}{\partial l_\parallel} \tag{9}$$

6 Solution methodology

$$\frac{k_1^2}{k_2^2}\hat{L}_T(\omega)\Phi + \hat{L}_P(\omega)\Phi - \sqrt{\frac{g_1}{g_2}}\frac{8\pi}{B^2 R}\frac{\partial P}{\partial L}\Phi = \sqrt{g_1}\frac{2\omega}{R}\frac{V_S}{V_A}\Theta, \quad (10)$$

$$\hat{L}_S(\omega)\Theta = \sqrt{g_1}\frac{2\omega}{R}\frac{V_S}{V_A}\Phi. \quad (11)$$

We solve on a certain L-shell (geostationary orbit, $L = 6.6$)

Boundary value problem: ω — parameter, k_1^2/k_2^2 — eigenvalue.

As a result, we find the dependence of the value k_1^2/k_2^2 on ω^2.

Plasma parameters:

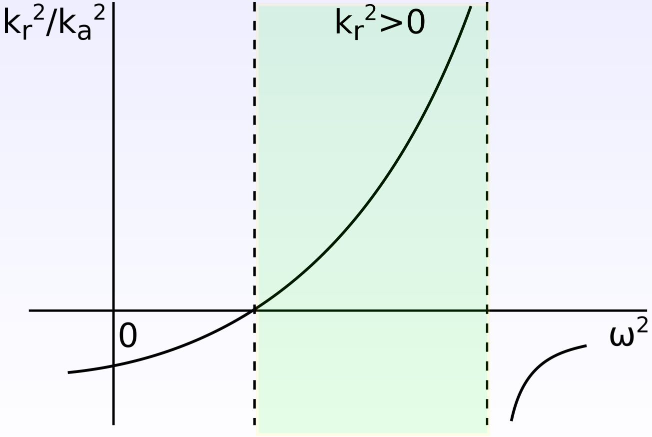
D — reverse pressure gradient at the equator, $|\nabla P| = P/D$

β — ratio of plasma pressure to magnetic pressure at the equator.

7 General view of the solution solution

Two transparency regions ($k_1^2 > 0$): one for the Alfven mode, the other for the SMS

Example of normal dispersion for SMS (wave, stability):



8 Changes in the structure of Θ_N during the transition from a stable to an unstable regime (from $k_r \to \infty$ to 0)

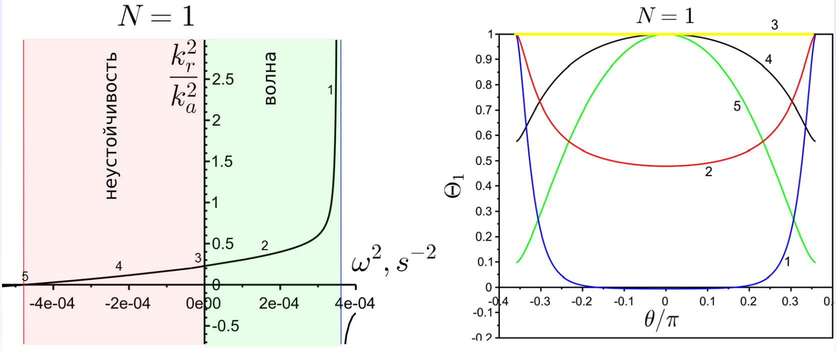


Рис.: 4.The ratio k_r^2/k_a^2 for the fundamental harmonic as a function of the square of the frequency ω^2. Changes in the structure of Θ_1 from magnetic latitude θ. Harmonics (1) and (2) correspond to the SMS modes ($\omega^2 > 0$). Harmonic (3) corresponds to the case $\omega^2 = 0$ (the instability boundary). Harmonics (4) and (5) correspond to the minimum and maximum instability increment.

9 Increment of ballooning instability

The ballooning instability increment $\gamma = \sqrt{-\omega^2}$ reaches its maximum value at $k_r = 0$, the ratio k_r/k_a tends to some critical value at $\omega \approx 0$.

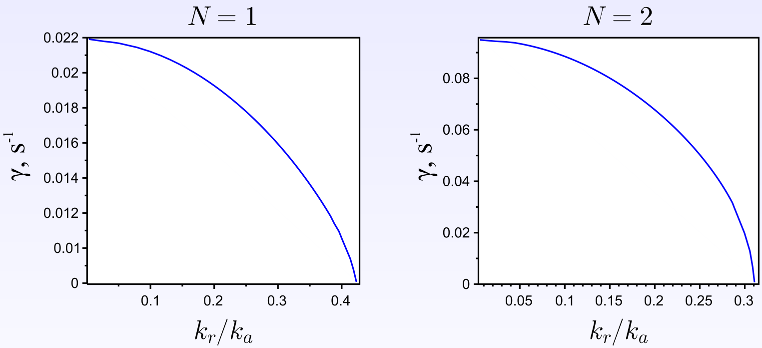


Рис.: 3. Dependence of the instability increment γ on the ratio of the radial and azimuthal wave numbers k_r/k_a for the fundamental $N = 1$ (left) and second $N = 2$ (right) harmonics.

10 Plasma stability

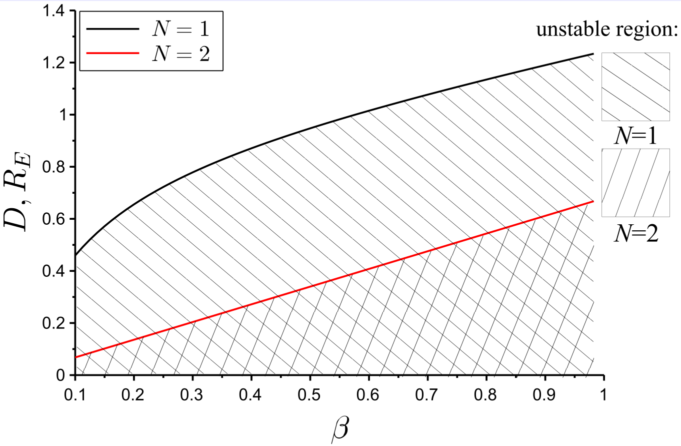


Рис.: 4.Instability threshold of the fundamental $N = 1$ (black) and second $N = 2$ (red) harmonics: scale of radial inhomogeneity of plasma pressure D depending on β at the equator. The instability region (on the $L = 6.6$ shell) lies below the corresponding curves.

11 Conclusion

- Ballooning instability can be considered as a continuation of the slow magnetosonic mode in the sense that it develops on the same dispersion branch. There is a significant difference in the eigenfunctions Φ and Θ in stable and unstable cases. In the stable regime, Θ and Φ are real. In the stable regime, Θ and Φ are real. In the unstable regime, both eigenfunctions cannot be simultaneously real: they have a phase shift of $\pi/2$.
- It has been established that the largest instability increment corresponds to the value $k_r^2 = 0$ and it approaches zero when the ratio k_r/k_a tends to some critical value near $\omega \approx 0$.
- The conditions of ballooning instability are considered depending on the radial non-uniformity of plasma pressure D and the parameter β. At large β the instability threshold is reached by smaller pressure gradients D. The situation with large β and small D is typical for magnetic storms, when a strong ring current develops in the magnetosphere.

Thank you attention !

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12 Unstable ballooning oscillations of the magnetic field

$$B_{\parallel} = \frac{ck_a}{\omega}\left[B\frac{V_S^2}{V_A^2}\nabla\cdot\xi + \frac{1}{\sqrt{g_1}}\frac{4\pi}{B^2}\frac{\partial P}{\partial L}\Phi\right] \text{ – compression component;} \quad (12)$$

$$B_r = -\frac{ck_a}{\omega}\frac{\partial\Phi}{\partial l} \text{ – radial component.} \quad (13)$$

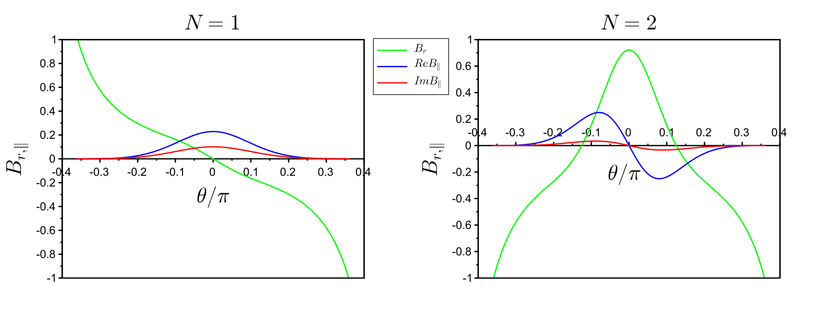


Рис.: 7. Unstable ballooning oscillations of the radial B_r and longitudinal $B_{\parallel}$ magnetic field for the fundamental (left) and second (right) harmonics at the maximum instability increment $(k_r = 0)$.

13 Basic equations

$$i\rho\omega^2\xi = -\nabla(\xi\cdot\nabla P + \gamma P\nabla\cdot\xi) - \frac{1}{4\pi}J_0\times\nabla\times[\xi\times B_0] + \\ + \frac{1}{4\pi}B_0\times\nabla\times[\nabla\times\xi\times B_0], \quad (14)$$

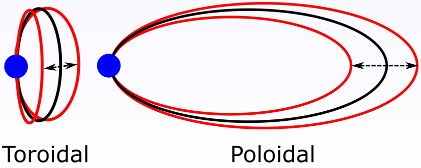
Φ — transverse potential of the electric field of the Alfvén wave:

$$E = E_\perp = -\nabla_\perp\Phi \quad (15)$$

Θ — proportional to the divergence of the plasma displacement:

$$\Theta = \frac{\sqrt{4\pi\gamma P}}{ck_a}\nabla\cdot\boldsymbol{\xi} \quad (16)$$

uniform plasma:
Φ — Alfvén mode
Θ — SMS mode
non-uniform plasma:
Φ — transverse electric field of the wave
Θ — rarefaction/compression of plasma



14 ВКБ–приближение

If the azimuthal number $k_a \gg \frac{1}{P}\frac{\partial P}{\partial L}$:

$$\Phi \propto \Phi_N(l_{\|}) \exp[i \int k_1 dL']$$

$$\frac{k_1^2}{k_2^2}\hat{L}_T(\omega)\Phi + \hat{L}_P(\omega)\Phi - \sqrt{\frac{g_1}{g_2}}\frac{8\pi}{B^2 R}\frac{\partial P}{\partial L}\Phi = \sqrt{g_1}\frac{2\omega}{R}\frac{V_S}{V_A}\Theta, \quad (17)$$

$$\hat{L}_S(\omega)\Theta = \sqrt{g_1}\frac{2\omega}{R}\frac{V_S}{V_A}\Phi. \quad (18)$$

$$V_A = \frac{B}{\sqrt{4\pi\rho}}, \quad V_S = \sqrt{\frac{\gamma P}{\rho}}, \quad V_C = \frac{V_A V_S}{\sqrt{V_A^2 + V_S^2}}. \quad (19)$$

R – radius of curvature of a field line

Mager P.N., Klimushkin D. Yu. Theory of azimuthally small-scale Alfvén waves in an axisymmeric magnetosphere with small but finite plasma pressure. JGR. 2002. Vol.107, P. 1356.

15 Basic equations: differential operators

Toroidal and poloidal differential operators:

$$\hat{L}_T(\omega) = \frac{\partial}{\partial l_{\|}}\sqrt{\frac{g_2}{g_1}}\frac{\partial}{\partial l_{\|}} + \sqrt{\frac{g_2}{g_1}}\frac{\omega^2}{V_A^2}, \qquad (20)$$

$$\hat{L}_P(\omega) = \frac{\partial}{\partial l_{\|}}\sqrt{\frac{g_1}{g_2}}\frac{\partial}{\partial l_{\|}} + \sqrt{\frac{g_1}{g_2}}\frac{\omega^2}{V_A^2}, \qquad (21)$$

These operators define the longitudinal structure of toroidal and poloidal Alfvén modes. The SMS operator:

$$\hat{L}_S(\omega) = \frac{\partial}{\partial l_{\|}}V_S^2\sqrt{g_1 g_2}\frac{\partial}{\partial l_{\|}} + \sqrt{g_1 g_2}\frac{V_S^2}{V_C^2}\omega^2. \qquad (22)$$

$$g_1 = \frac{\cos^6\theta}{1+3\sin^3\theta}, \;\; g_2 = L^2\cos^6\theta. \qquad (23)$$

$$\partial l_{\|} = \sqrt{g_3}\partial x^3 = L\cos\theta\sqrt{1+3\sin^3\theta}\partial\theta. \qquad (24)$$

16 Boundary conditions on the ionosphere

Characteristics of the ionosphere:

$$\Phi(l_{\|}) = \mp i \frac{c^2}{4\pi\omega\Sigma_{\pm}} \frac{\partial\Phi}{\partial l_{\|}} \tag{25}$$

Ideal conductivity:

$$\Phi|_{\theta=\theta_{\pm}} = 0. \tag{26}$$

$$\frac{\omega^2}{V_S^2} \frac{\omega}{k_a c} = -\frac{\omega}{\sqrt{4\pi\gamma P}} \frac{\partial\Theta}{\partial l_{\|}} \tag{27}$$

High density, $\xi_{\|} = 0$:

$$\left.\frac{\partial\Theta}{\partial l_{\|}}\right|_{\theta=\theta_{\pm}} = 0. \tag{28}$$