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ФОРМУЛА БРЕЙТ–ВИГНЕРА ДЛЯ ФЕРМИОНОВ

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BREIT–WIGNER FORMULA FOR FERMIONS

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Мы обсуждаем процедуру получения теоретико-полевого аналога формулы Брейт–Вигнера для фермионов.

We discuss the procedure of obtaining the quantum field theory analog of Breit–Wigner formula for fermions.

1. Introduction

The Breit–Wigner formula describing the processes of production and decay of unstable particles is widely used in hadron and nuclear physics. The original formula [1], which was applied to scattering of slow neutrons, is a non-relativistic one and, besides, the parameters (mass and width) are supposed to be energy-independent. It is clear that this approximation works well only for very narrow states; so description of real hadron resonances (especially with improving the experimental accuracy) needs more adequate methods based on quantum field theory (QFT). In principal it is a well-known procedure (see, e.g. [2]), though different techniques can be used.

2. Boson resonance in QFT

Let us consider first a more evident case of the boson resonance. The unstable particle is usually associated with the Breit-Wigner formula for an amplitude

$$M(a \to R \to b) = g_a \cdot \frac{1}{M_R^2 - s - i\Gamma_R M_R} \cdot g_b. \qquad (1)$$

Here the factor $G(s) = 1/(M_R^2 - s - i\Gamma_R M_R)$ represents relativistic propagator of unstable particle.

The similar formula may be obtained in framework of quantum field theory by means of Dyson summation of the self-energy insertions into propagator. Equivalently, we should solve the Dyson–Schwinger equation for full non-renormalized propagator

$$G = G_0 + GJG_0. \qquad (2)$$

Here $G_0 = (M^2 - s)^{-1}$ and $G(s)$ are free and full propagators respectively, $J(s)$ is the self-energy contribution.

The equation (2) may be rewritten in terms of inverse propagators $S(s) = G^{-1}(s)$ [1]

$$S = S_0 - J = M^2 - s - J(s). \qquad (3)$$

If one use the on-mass-shell scheme of renormalization, then M in full propagator is the renormalized mass and one should subtract the self-energy contribution twice at this point

$$S^r = M^2 - s - [J(s) - \operatorname{Re} J(M^2) - \operatorname{Re} J'(M^2)(s - M^2)]. \qquad (4)$$

After this we have similar to (1) formula but with "running" mass and width. From QFT point of view the Breit-Wigner formula is rather rough approach, when we neglect the energy dependence in mass and width (i.e. the energy dependence in loop contribution $J(s)$) and which may be adequate only for narrow resonance $\Gamma_R \to 0$ far from threshold.

3. Fermion resonance in QFT

The fermion propagator dressing looks rather similar

$$G_0 = \frac{1}{\hat{p} - M} \Rightarrow G = \frac{1}{\hat{p} - M - \Sigma(p)},$$

where $\Sigma(p) = A(p^2) + \hat{p}B(p^2)$ is a self-energy contribution.

To renormalize the obtained expression it is convenient to work with the inverse propagator $S(p) = G^{-1}(p)$. Usually the renormalization condition is formulated as the decomposition of inverse propagator in terms of $\hat{p} - M$

$$S(p) = \hat{p} - M + o(\hat{p} - M) \text{ at } \hat{p} \to M. \qquad (5)$$

We use convenient basis consisting off-shell projection operators $\Lambda^{\pm}$

$$\Lambda^{\pm} = \frac{1}{2}\left(1 \pm \frac{\hat{p}}{E}\right), \; E = \sqrt{p^2}, \qquad (6)$$

which simplifies significantly all the γ-matrix operations and makes more evident the renormalization condition. To illustrate let us write down the self-energy contribution in this basis.

$$\begin{aligned}\Sigma(p) &= A(p^2) + \hat{p}B(p^2) = \\ &= \Lambda^+\left(A(E^2) + EB(E^2)\right) + \Lambda^-\left(A(E^2) - EB(E^2)\right).\end{aligned} \qquad (7)$$

One can see that coefficients $\Sigma^{\pm}$ in this basis are bound by simple substitution $\Sigma^-(E) = \Sigma^+(-E)$ and coefficients of the full propagator have this property also.

[1] If we consider the self-energy $J(s)$ as a known function (i.e. neglecting the vertex dressing) then we come to the so called *rainbow* approximation, see e.g. review [3**Ошибка! Источник ссылки не найден.**]. This approximation is sufficient to respect the analytical properties and unitarity and is widely used in resonance physics (sometimes as some post-rainbow approximation).

The reversing of propagator is very easy due to simple multiplicative properties of the basis. If the inverse propagator has a decomposition

$$S(p) = \Lambda^+ S^+ + \Lambda^- S^-, \tag{8}$$

with symmetry property $\cancel{S}^-(E) = S^+(-E)$, then the propagator $G(p)$ has form

$$G(p) = \Lambda^+ \frac{1}{S^+(E)} + \Lambda^- \frac{1}{S^-(E)}. \tag{9}$$

So, the explicit form of the full un-renormalized propagator is evident

$$\begin{gathered} G(p) = \Lambda^+ \frac{1}{E - M - (A(E^2) + EB(E^2))} + \\ + \Lambda^- \frac{1}{-E - M - (A(E^2) - EB(E^2))}. \end{gathered} \tag{10}$$

Thus, using the Λ-basis we have separated the γ-matrix structure and should renormalize the scalar coefficients dependent on E. More precisely, we should renormalize only the G^+ component, while the other coefficient G^- will be obtained after that by substitution $E \to -E$.

If one say about bound state located below the threshold, the renormalization leads to the following condition for the self-energy contribution:

$$S^+ = E - M + o(E - M) \text{ at } E \to M$$

One can convince yourself that the final expression obtained with using the projection operator basis coincides with the standard one presented in any textbook.

If we deal with a resonance located higher the threshold, the renormalization condition takes the form

$$S^+ = E - M + o(E - M) + \tfrac{i}{2}\Gamma \text{ at } E \to M. \tag{11}$$

Note that real part of (11) is some requirement on the subtraction constants of self-energy functions $A(p^2), B(p^2)$

$$\begin{gathered} \operatorname{Re} A(M^2) + M \operatorname{Re} B(M^2) = 0, \\ 2M \operatorname{Re} A'(M^2) + \operatorname{Re} B(M^2) + 2M^2 \operatorname{Re} B'(M^2) = 0, \end{gathered} \tag{12}$$

whereas the imaginary part of the condition (11) simply relates the coupling constant with width[2].

$$\operatorname{Im} A(M^2) + \operatorname{Im} B(M^2) = -\frac{\Gamma}{2}. \tag{13}$$

Eq. (12) fixes the loop subtraction constants and after that functions $A(E^2), B(E^2)$ are defined completely. The inverse propagator may be written in form similar to Breit-Wigner formula (1) but with "running" mass and width

$$S^+ = E - M(E) + \tfrac{i}{2}\Gamma(E). \tag{14}$$

Another component is obtained by the substitution $E \to -E$

$$S^-(E) = -E - M(-E) + \tfrac{i}{2}\Gamma(-E). \tag{15}$$

If one look at the self-energy contribution, one can see that there are the symmetric and antisymmetric contributions under the $E \to -E$ exchange. Therefore the running mass and width also may be divided into two parts

$$M(E) = M^S(E) + M^A(E), \; \Gamma(E) = \Gamma^S(E) + \Gamma^A(E) \tag{16}$$

and components of renormalized propagator take the form

$$\begin{gathered} S^+ = E - M^S(E) - M^A(E) + \frac{i}{2}\left(\Gamma^S(E) + \Gamma^A(E)\right), \\ S^- = -E - M^S(E) + M^A(E) + \frac{i}{2}\left(\Gamma^S(E) - \Gamma^A(E)\right). \end{gathered} \tag{17}$$

Let us stress that these components are normalized at different points

$$S^+ \approx E - M + \tfrac{i}{2}\Gamma \text{ at } E \to M$$

$$S^+ \approx -E - M + \tfrac{i}{2}\Gamma \text{ at } E \to -M$$

Returning from projection operators to $\hat{p}, I$ basis, we obtain the following formula for the resonance propagator

$$\begin{gathered} G(p) = \frac{1}{\Delta}\Big[M^S(E) - \tfrac{i}{2}\Gamma^S(E) + \\ + \hat{p}\left(1 - \frac{M^A(E) - \tfrac{i}{2}\Gamma^A(E)}{E}\right)\Big], \end{gathered} \tag{19}$$

where

$$\begin{gathered} \Delta(E^2) = \left(E - M^A(E) + \tfrac{i}{2}\Gamma^A(E)\right)^2 - \\ - \left(M^S(E) - \tfrac{i}{2}\Gamma^S(E)\right)^2 \end{gathered}. \tag{20}$$

Let us compare it with boson Breit–Wigner formula for the inverse propagator $S(p^2)$ written in a similar form

$$\begin{gathered} S(s) = s - M^2(s) + iM\Gamma(s) \approx \\ \approx E^2 - (M - \tfrac{i}{2}\Gamma)^2 = [E - (M - \tfrac{i}{2}\Gamma)][E + (M - \tfrac{i}{2}\Gamma)]. \end{gathered} \tag{21}$$

One can see that the fermion denominator in (20) turns into this expression in absence of antisymmetric contributions $M^A = \Gamma^A = 0$.

Let us compare also the interference pictures near the positive energy pole in elastic amplitude.

Boson case

[2] It is known that the normalization on the pole in complex energy plane is preferable from theoretical point of view [4**Ошибка! Источник ссылки не найден.**, 5**Ошибка! Источник ссылки не найден.**] but for our purpose the more crude recipe (11) is sufficient.

$$T = \frac{g^2}{M^2 - s - iM\Gamma} = = -\frac{g^2}{2M}\left(\frac{1}{E - M + \frac{i}{2}\Gamma} + \frac{1}{-E - M + \frac{i}{2}\Gamma}\right). \quad (22)$$

Fermion case

For definiteness let us consider the process $\pi N \to R(1/2^+) \to \pi N$ and look at the CMS helicity amplitude M_{++} at $E \to M$

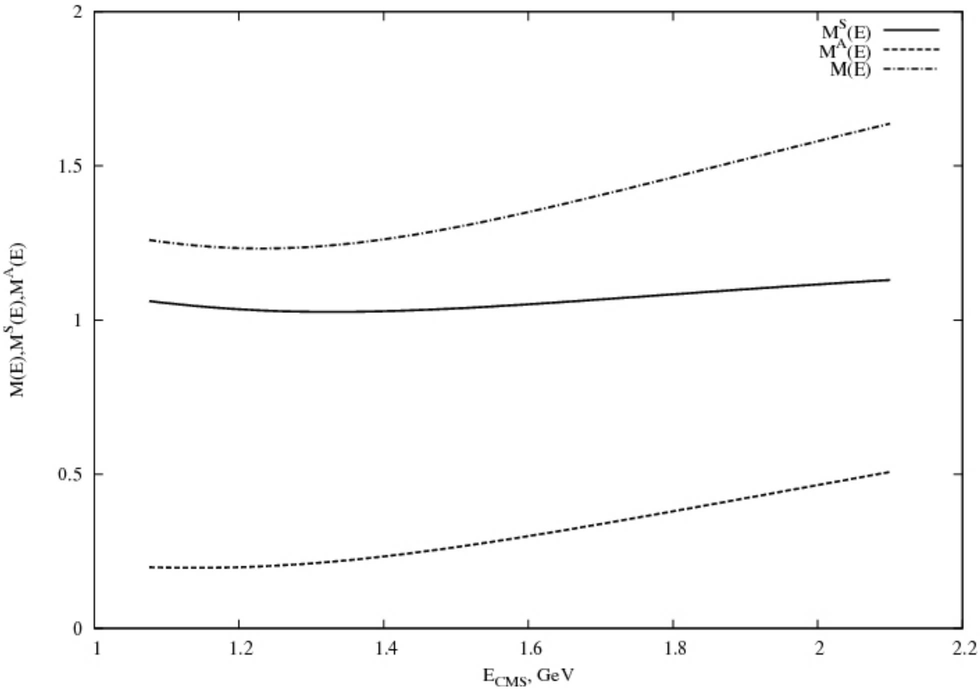

Fig. 1. “Running” mass $M(E)$ and symmetric and antisymmetric masses $M^S(E)$, $M^A(E)$.

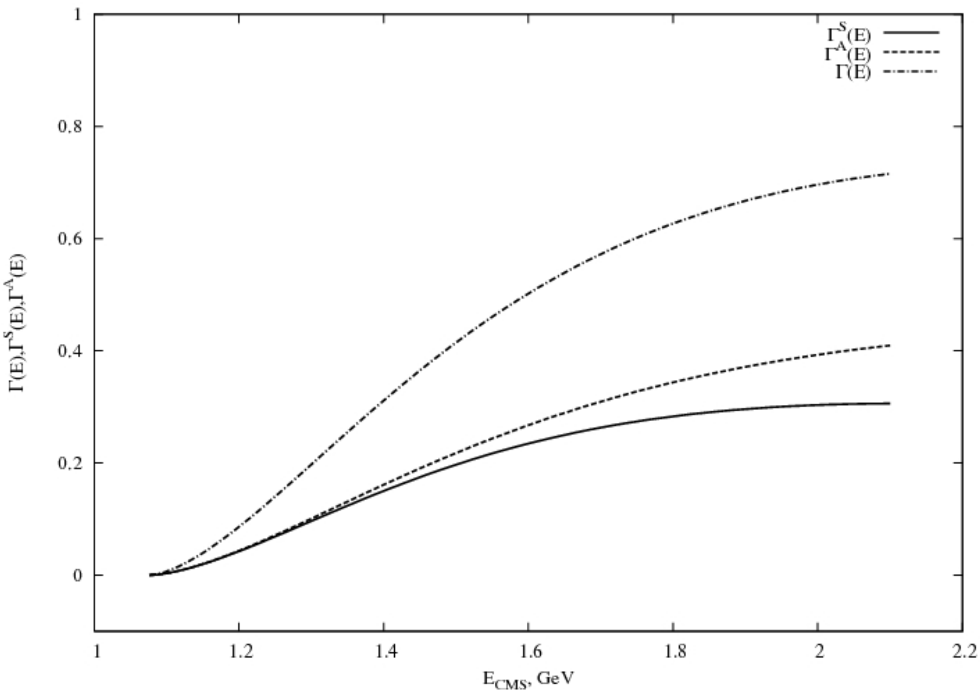

Fig. 2. “Running” width $\Gamma(E)$ and symmetric and antisymmetric widths $\Gamma^S(E)$, $\Gamma^A(E)$.

$$M_{++} = \cos\frac{\vartheta}{2}\Bigg[(p^0+m)\frac{g^2}{-E-(M^S-M^A)+\frac{i}{2}(\Gamma^S-\Gamma^A)} -$$
$$-(p^0-m)\frac{g^2}{E-M+\frac{i}{2}\Gamma}\Bigg]. \tag{23}$$

where p^0 is the nucleon CM energy.

One can see that in contrast to boson case the background contribution in vicinity of $E = M$ is not expressed in terms of M and Γ. The observed specific in fermion Breit–Wigner formula (presence of antisymmetric in E terms $M^A(E)$, $\Gamma^A(E)$) is generated by $\hat{p}$ contribution in self-energy $\Sigma(p) = A(p^2) + \hat{p}B(p^2)$. As a result instead of two parameters (M and Γ) the fermion Breit–Wigner formula is described by four parameters. But this difference may be essential only for broad resonance since in the limit $\Gamma \to 0$ the influence of background is negligible.

Let us illustrate the Breit–Wigner formula for fermions by considering Δ-resonance. The full spin 3/2 propagator has form [6]

$$G_{\mu\nu} = \Lambda^+ P^{3/2}_{\mu\nu}\frac{1}{E-M-\overline{J}_1(E)} +$$
$$+\Lambda^- P^{3/2}_{\mu\nu}\frac{1}{-E-M-\overline{J}_1(-E)}.$$

Fig. 1 shows the “running” mass $M(E)$, symmetric and antisymmetric masses $M^S(E)$, $M^A(E)$ respectively. Fig. 2 shows the “running” width $\Gamma(E)$, symmetric and antisymmetric widths $\Gamma^S(E)$, $\Gamma^A(E)$ respectively.

4. Conclusions

We have discussed in detail obtaining the Breit–Wigner-like formula in quantum field theory for fermion resonance. It has own specifics as compared with boson resonance case where even the resonance denominator does not coincide with its boson analog. We have found that fermion resonance is not described by two parameters, i.e. mass and width but needs at least four parameters. Therefore using some simplified formulae for experiment description should be made carefully.

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